Preprint: July 27, 2004.
Submitted to Contr. Gen. Alg. 16

ON VARIETIES OF SEMIGROUPS OF RELATIONS WITH OPERATIONS OF CYLINDRIFICATION IN: CONTRIBUTIONS TO GENERAL ALGEBRA, V. 16, P. 1–6, 2005

D.A.BREDIKHIN

ABSTRACT. The finite basis of identities for the varieties generated by semigroups of relations with operations of cylindrification is found.

In the investigation of algebras of relations, one of the most important problem is to study their identities [1,2,3]. For any set Ω of operations on binary relations, denote by $R\{\Omega\}$ the class of algebras whose elements are binary relations and whose operations are members of Ω. Let $Var\{\Omega\}$ be the variety generated by $R\{\Omega\}$.

We shall consider the operations of relation product $\circ$ and two unary operations of cylindrification [4] defined as follows:

$$\nabla_1(\rho) = \{(x,y) : (\exists z)(x,z) \in \rho\}, \quad \nabla_2(\rho) = \{(x,y) : (\exists z)(z,y) \in \rho\}.$$

It is well-known that the variety $Var\{\circ\}$ is equal to the class of all semigroups. The following theorems give the finite basis of identities for the varieties $Var\{\circ, \nabla_1\}$ and $Var\{\circ, \nabla_2\}$.

Theorem 1. *An algebra $(A, \cdot, ^*)$ of the type $(2,1)$ belongs to the variety $Var\{\circ, \nabla_1\}$ if and only if it satisfies the identities:*

$$(xy)z = x(yz) \ (1), \quad (x^*)^* = x^* \ (2), \quad (x^*)^2 = x^* \ (3), \ (xy)^* = xy^* \ (4),$$
$$xy^*x^* = xy^* \ (5), \quad x^*y^*z^* = x^*z^*y^* \ (6), \ x^*y^*zy = x^*zy \ (7).$$

Theorem 2. *An algebra $(A, \cdot, ^*)$ of the type $(2,1)$ belongs to the variety $Var\{\circ, \nabla_2\}$ if and only if it satisfies the identities (1)-(3) and*

$$(xy)^* = x^*y, \quad x^*y^*x = y^*x, \ x^*y^*z^* = y^*x^*z^*, \ xyx^*z^* = xyz^*.$$

We shall prove Theorem 1. Theorem 2 can be proved analogously. Let us divide the consequent proof into four steps.

Step 1. The proof is based on the result of [5]. First, let us give some definitions and notations to formulate this result and for use in the proof.

2000 *Mathematics Subject Classification.* Primary 20M07; secondary 20M20.
Key words and phrases. Semigroups, relations.

Let $Rel(U)$ be the set of all binary relations on U. To any first order formula $\phi(z_0, z_1, r_1, \ldots, r_m)$ having m binary predicate symbols $r_1, \ldots, r_m$ and two free individual variables z_0, z_1, we can associate an m-ary operation F_φ on $Rel(U)$ defined as follows:

$$F_\varphi(R_1, \ldots, R_m) = \{(x, y) \in U \times U : \varphi(x, y, R_1, \ldots, R_m)\},$$

where $\varphi(x, y, R_1, \ldots, R_m)$ means that the formula φ holds whenever z_0, z_1 are interpreted as x, y and $r_1, \ldots, r_m$ are interpreted as relations $R_1, \ldots, R_m \in Rel(U)$.

An operation on binary relations is called primitive positive [6] if it can be defined by a first order formula in which only existential quantifiers and conjunctions appear. Primitive positive operations can be described by using graphs [6].

Denote by N the set of all natural numbers. A labeled graph is a pair $G = (V,\ E)$, where $V = V(G)$ is a finite set, called a vertex set, and $E = E(G) \subset V \times N \times V$ is a ternary relation. A triple $(u, k, v) \in E$ is called an edge from u to v labeled by k, and it will graphically represented by $u\cdot \stackrel{k}{\to} \cdot v$.

By an input-output-pointed labeled graph we mean a structure $G = (V, E, in, out)$, where (V, E) is a labeled graph; $in = in(G)$ and $out = out(G)$ are two distinguished vertices (not necessarily different) called input and output vertex respectively.

In what follows, we shall usually speak simply graphs if it does not leads to a confusion. An isomorphism of graphs is defined by usual way. All graphs will be considered up to isomorphism.

Let $F = F_\varphi$ be a primitive positive operation determined by a formula φ. Then the input-output-pointed labeled graph $G = G(F) = G(\varphi)$ associated with F is defined as follows (see [6]): $V(G)$ is the set of all subscripts of individual variables of φ; $in(G) = 0$, $out(G) = 1$; $(i, k, j) \in E(G)$ if and only if the atomic formula $r_k(z_i, z_j)$ occurs in φ; if the formula $z_i = z_j$ occurs in φ, then vertices i and j are identified.

Note that the graphs corresponding to the operations of relation product $\circ$ and cylindrification ∇_1, ∇_1 are the following:

$$in\cdot \stackrel{1}{\to} \cdot \stackrel{2}{\to} \cdot out\ ; \quad in\cdot \stackrel{1}{\to} \cdot \quad \cdot\, out\ ; \quad in\, \cdot \quad \cdot \stackrel{1}{\to} \cdot out$$

Let $G = (V, E, in, out)$ and $G_k = (V_k, E_k, in_k, out_k)$ $(k = 1, \ldots, m)$ be graphs with pairwise disjoint vertex sets. The composition $G(G_1, \ldots, G_m)$ is the graph constructed as follows [6]: take G and substitute every edge $(u, k, v) \in E$ by the graph G_k identifying the input vertex in_k with u and the output vertex out_k with v.

Let $\Omega = \{F_{\varphi_1}, \ldots, F_{\varphi_n}\}$, $G_1 = G(\varphi_1), \ldots, G_n = G(\varphi_n)$, and $A = (A, f_1, \ldots, f_n)$ be an universal algebra of the corresponding type. For any term p of A define the graph $G(p) = (V_p, E_p, in(p), out(p))$ in the following way:

1) if $p = x_k$, then $G(p)$ is the following: $in\cdot \xrightarrow{k} \cdot out$;

2) if $p = f_k(p_1, \dots, p_m)$, then $G(p)$ is the composition $G_k(G(p_1), \dots, G(p_m))$.

Given two graphs $G_1 = (V_1, E_1, in_1, out_1)$ and $G_2 = (V_2, E_2, in_2, out_2)$, a mapping $f : V_2 \to V_1$ is called a homomorphism from G_2 to G_1 if $f(in_2) = in_1$, $f(out_2) = out_1$, and $(f(u), k, f(v)) \in E_1$ whenever $(u, k, v) \in E_2$. We write that $G_1 \prec G_2$ if there exists a homomorphism from G_2 to G_1.

Let $Eq\{\Omega\}$ be the equational theory of $R\{\Omega\}$. Now, we can formulate the main result of [5]:

(**Th.1**) *The identity $p = q$ belongs to the equational theory $Eq\{\Omega\}$ if and only if $G(p) \prec G(q)$ and $G(q) \prec G(p)$.*

Step 2. Denote by Σ the equational theory of algebras that satisfy the identities (1)-(7), and let Ξ be the set of all terms of algebras $(A, \cdot, ^*)$ of the type $(2, 1)$. For any p_1 and p_2 from Ξ, we shall write $p_1 \cong p_2$ whenever the identity $p_1 = p_2$ belongs to Σ.

Let Λ be the set of all words on the alphabet $\{x_1, \dots, x_n, \dots\}$, $\odot$ be the empty word, and $\tilde{\Lambda} = \Lambda \setminus \{\odot\}$.

Lemma 1. *For any term $p \in \Xi$ there exist $\alpha_0, \alpha_1, \dots, \alpha_n$ $(n \geq 0)$ such that $p \cong (\alpha_1)^* \dots (\alpha_n)^* \alpha_0$, where $\alpha_1, \dots, \alpha_n \in \tilde{\Lambda}$, $\alpha_0 \in \Lambda$, and $\alpha_0 \neq \odot$ whenever $n = 0$.*

Proof. The proof is by induction on the definition of p. It is clear for $p = x_k$. Suppose that $p \cong (\alpha_1)^* \dots (\alpha_n)^* \alpha_0$. If $\alpha_0 \neq \odot$, then using (4), we have

$$(p)^* = ((\alpha_1)^* \dots (\alpha_n)^* \alpha_0)^* \cong (\alpha_1)^* \dots (\alpha_n)^* (\alpha_0)^*.$$

If $\alpha_0 = \odot$, then by (2) and (4), we have $(p)^* = ((\alpha_1)^* \dots (\alpha_n)^*)^* \cong (\alpha_1)^* \dots ((\alpha_n)^*)^* \cong (\alpha_1)^* \dots (\alpha_n)^*$.

Further, suppose that $p_1 \cong (\alpha_1)^* \dots (\alpha_n)^* \alpha_0$ and $p_2 \cong (\beta_1)^* \dots (\beta_m)^* \beta_0$.

If $m = 0$, then using (1), we get

$$p_1 p_2 = ((\alpha_1)^* \dots (\alpha_n)^*)((\beta_1)^* \dots, (\beta_m)^* \beta_0) \cong (\alpha_1)^* \dots (\alpha_n)^* (\beta_1)^* \dots, (\beta_m)^* \beta_0.$$

If $m > 0$, then using (1) and (4), we have

$$p_1 p_2 = ((\alpha_1)^* \dots (\alpha_n)^* \alpha_0)((\beta_1)^* \dots (\beta_m)^* \beta_0) \cong$$
$$(\alpha_1)^* \dots (\alpha_n)^* \alpha_0 (\beta_1)^* \dots (\beta_m)^* \beta_0 \cong (\alpha_1)^* \dots (\alpha_n)^* (\alpha_0 \beta_1)^* \dots (\beta_m)^* \beta_0.$$

□

Lemma 2. *Let $p = (\alpha_1)^* \dots (\alpha_n)^* \alpha_0$, $\alpha_k = \beta_1 \beta \beta_2$ for some $k \in [0, n]$, $\beta_1, \beta_2 \in \Lambda$, $\beta \in \tilde{\Lambda}$, and $n > 0$ whenever $k = 0$. Then*

$$p \cong (\alpha_1)^* (\alpha_2)^* \dots (\alpha_n)^* (\beta)^* \alpha_0.$$

Proof. Suppose that $k > 0$. If $\beta_2 = \odot$, then by (2) and (4), we get $(\alpha_k)^* = (\beta_1 \beta)^* \cong \beta_1 (\beta)^* \cong \beta_1 (\beta)^* (\beta)^* \cong (\beta_1 \beta)^* (\beta)^* = (\alpha_k)^* (\beta)^*$. If $\beta_2 \neq \odot$, then

using (4) and (5), we have $(\alpha_k)^* = (\beta_1\beta\beta_2)^* \cong \beta_1\beta(\beta_2)^* \cong \beta_1\beta(\beta_2)^*(\beta)^* \cong (\beta_1\beta\beta_2)^*(\beta)^* = (\alpha_k)^*(\beta)^*$. Therefore, by (6), we obtain
$p = (\alpha_1)^* \dots (\alpha_k)^* \dots (\alpha_n)^*\alpha_0 \cong (\alpha_1)^* \dots (\alpha_k)^*(\beta)^* \dots (\alpha_n)^*\alpha_0 \cong$
$(\alpha_1)^* \dots (\alpha_n)^*(\beta)^*\alpha_0$.

Suppose that $k = 0$. If $\beta_1 = \odot$, then using (2), (6), (7), we have
$p = (\alpha_1)^* \dots (\alpha_n)^*\alpha_0 = (\alpha_1)^* \dots (\alpha_n)^*\beta\beta_2 \cong (\alpha_1)^* \dots (\alpha_n)^*(\alpha_n)^*\beta\beta_2 \cong$
$(\alpha_1)^* \dots (\alpha_n)^*(\beta)^*(\alpha_n)^*\beta\beta_2 \cong (\alpha_1)^* \dots (\alpha_n)^*(\alpha_n)^*(\beta)^*\beta\beta_2 \cong$
$(\alpha_1)^* \dots (\alpha_n)^*(\beta)^*\alpha_0$. If $\beta_1 \neq \odot$, then by (7), we obtain
$p = (\alpha_1)^* \dots (\alpha_n)^*\alpha_0 = (\alpha_1)^* \dots (\alpha_n)^*\beta_1\beta\beta_2 \cong (\alpha_1)^* \dots (\alpha_n)^*(\beta)^*\beta_1\beta\beta_2 =$
$(\alpha_1)^* \dots (\alpha_n)^*(\beta)^*\alpha_0$. □

Step 3. According to the definition, the graph $G(p) = (V_p, E_p, in(p), out(p))$ for $p \in \Xi$ can be constructed in the following way.

Let $p = \alpha = x_{i_1}x_{i_2}\dots x_{i_n} \in \tilde{\Lambda}$. Then
$V_p = V_\alpha = \{v_0, \dots, v_n\}$, $E_p = E_\alpha = \{(v_{k-1}, i_k, v_k) : k \in [1, n]\}$,
and $in(p) = in(\alpha) = v_0$, $out(p) = out(\alpha) = v_n$:

$$in(\alpha) = v_0 \cdot \overset{i_1}{\to} \cdot \overset{i_2}{\to} \cdot \dots \cdot \overset{i_n}{\to} \cdot v_n = out(\alpha)$$

Let $p = (\alpha)^*$. Then $V_p = V_{\alpha^*} = V_\alpha \cup \{v_{n+1}\}$, $E_p = E_{\alpha^*} = E_\alpha$,
and $in(p) = in(\alpha^*) = in(\alpha) = v_0$, $out(p) = out(\alpha^*) = v_{n+1}$:

$$in(\alpha^*) = v_0 \cdot \overset{i_1}{\to} \cdot \overset{i_2}{\to} \cdot \dots \cdot \overset{i_n}{\to} \cdot \;\; \cdot v_{n+1} = out(\alpha^*)$$

Let $p = (\alpha_1)^*(\alpha_2)^* \dots, (\alpha_n)^*\alpha_0$ and $n > 1$. We shall assume that the sets $V_{\alpha_1^*}, \dots, V_{\alpha_n^*}, V_{\alpha_0}$ are pairwise disjoint.

If $\alpha_0 = \odot$, then $V_p = V_{\alpha_1} \cup \dots \cup V_{\alpha_{n-1}} \cup V_{\alpha_n^*}$, $E_p = E_{\alpha_1} \cup \dots \cup E_{\alpha_{n-1}} \cup E_{\alpha_n}$, and $in(p) = in(\alpha_1)$, $out(p) = out(\alpha_n^*)$.

If $\alpha_0 \neq \odot$, then $V_p = V_{\alpha_1} \cup \dots \cup V_{\alpha_n} \cup V_{\alpha_0}$, $E_p = E_{\alpha_1} \cup \dots \cup E_{\alpha_n} \cup E_{\alpha_0}$, and $in(p) = in(\alpha_1)$, $out(p) = out(\alpha_0)$.

Lemma 3. *Suppose that $\alpha, \beta \in \tilde{\Lambda}$ and there exists a map f from V_β to V_α such that $(f(u), k, f(v)) \in E_\alpha$ whenever $(u, k, v) \in E_\beta$. Then there exist $\beta_1, \beta_2 \in \Lambda$ such that $\alpha = \beta_1\beta\beta_2$.*

Proof. Let $\alpha = x_{i_1}x_{i_2}\dots x_{i_n}$, $\beta = x_{j_1}x_{j_2}\dots x_{j_m}$, and $V_\alpha = \{v_0, \dots, v_n\}$, $V_\beta = \{v'_0, \dots, v'_m\}$. Suppose that $f(v'_0) = v_l$. Then, it is easy to see that $f(v'_k) = v_t$ and $x_{j_k} = x_{i_t}$, where $t = l+k$. Therefore, it is sufficiently to put $\beta_1 = x_{i_1}\dots x_{i_s}$ (or $\beta_1 = \odot$ whenever $l = 1$) and $\beta_2 = x_{i_w}\dots x_{i_n}$ (or $\beta_1 = \odot$ whenever $l + m = n$), where $s = l$ and $w = l + m + 1$. □

Step 4. It is clear to check that the operations $\circ$ and ∇_1 satisfy the identities (1)-(7). It follows that $Eq\{\circ, \nabla_1\} \subset \Sigma$. Therefore, to prove Theorem 1 it is sufficiently to show that $\Sigma \subset Eq\{\circ, \nabla_1\}$.

Let us assume that the identity $p_1 = p_2$ belongs to $Eq\{\circ, \nabla_1\}$. According to the result (**Th.1**) of [5], we have $G(p_1) \prec G(p_2)$ and $G(p_2) \prec G(p_1)$, i.e., there exist the homomorphisms f from $G(p_2)$ to $G(p_1)$ and g from $G(p_1)$ to $G(p_2)$. By Lemma 1, we can assume that $p_1 = (\alpha_1)^* \dots (\alpha_n)^* \alpha_0$ and $p_2 = (\beta_1)^* \dots (\beta_m)^* \beta_0$.

Suppose that $\alpha_0 \neq \odot$. Then there exists an edge in $G(p_1)$ leads to $out(p_1)$. It follows that there exists an edge in $G(p_2)$ leads to $g(out(p_1)) = out(p_2)$, hence, $\beta_0 \neq \odot$. Analogously, $\beta_0 \neq \odot$ implies $\alpha_0 \neq \odot$.

Suppose that $n = 0$ and $m > 0$. Since $n = 0$, we have $p_1 = \alpha_0$. Then according to the structure of the graph $G(\alpha_0)$ there exists a path in $G(p_1)$ from $in(p_1)$ to $out(p_1)$. Hence, there exists a path in $G(p_2)$ from $g(in(p_1)) = in(p_2)$ to $g(out(p_1)) = out(p_2)$. This contradicts to the structure of the graph $G(p_2)$. Therefore, $m > 0$ implies $n > 0$. Analogously, $n > 0$ implies $m > 0$.

Let us assume that $\alpha_0 \neq \odot$ and $\beta_0 \neq \odot$. Since $f(out(p_2)) = f(out(\beta_0)) = out(p_1) = out(\alpha_0)$ and $g(out(p_1)) = g(out(\alpha_0)) = out(p_2) = out(\beta_0)$, we have $f(V_{\beta_0}) \subset V_{\alpha_0}$ and $g(V_{\alpha_0}) \subset V_{\beta_0}$. From Lemma 3 it follows that $\alpha_0 = \beta_0$.

Suppose that $n, m > 0$. Since $f(in(p_2)) = f(in(\beta_1)) = in(p_1) = out(\alpha_1)$ and $g(in(p_1)) = g(in(\alpha_1)) = in(p_2) = in(\beta_1)$, we have $f(V_{\beta_1)} \subset V_{\alpha_1}$ and $g(V_{\alpha_1}) \subset V_{\beta_1}$. From Lemma 2 it follows that $\alpha_1 = \beta_1$. Further, for any $k = 2, \dots, n$, we have $f(V_{\beta_k}) \subset V_{\alpha_{k'}}$ and $g(V_{\alpha_k}) \subset V_{\beta_{k''}}$ for some k', k''. From Lemma 3 it follows that $\alpha_{k'} = \beta'_k \beta_k \beta''_k$ and $\beta_{k''} = \alpha'_k \alpha_k \alpha''_k$ for some α'_k, α''_k, β'_k, β''_k from Λ. Hence, using Lemma 2 and identity (6), we obtain $p_1 = (\alpha_1)^*(\alpha_2)^* \dots (\alpha_n)^* \alpha_0 \cong (\alpha_1)^*(\alpha_2)^* \dots (\alpha_n)^*(\beta_2)^* \dots (\beta_m)^* \alpha_0 =$ $(\beta_1)^*(\alpha_2)^* \dots (\alpha_n)^*(\beta_2)^* \dots (\beta_m)^* \beta_0 \cong (\beta_1)^*(\beta_2)^* \dots (\beta_m)^*(\alpha_2)^* \dots (\alpha_n)^* \beta_0 \cong$ $(\beta_1)^*(\beta_2)^* \dots (\beta_m)^* \beta_0 = p_2$.

Therefore, the identity $p_1 = p_2$ belongs to Σ, i.e., $\Sigma \subset Eq\{\circ, \nabla_1\}$. This completes the proof of Theorem 1.

References

[1] A. Tarski.: *On the calculus of relations.* J. Symbolic Logic, **6**(1941), 73-89.

[2] A. Tarski.: Some methodological results concerning the calculus of relations. J. Symbolic Logic, **18**(1953), 188-189.

[3] B. M. Schein.: Relation algebras and function semigroups. Semigroup Forum, **1** (1970), 1-62.

[4] L. Henkin, J. D. Monk, A. Tarski.: Cylindric algeras I, II. North-Holland, Amsterdam, 1971, 1985.

[5] D. A. Bredikhin.: The equational theory of algebras of relations with positive operations. Izv. vuzov, Matematika, **3** (1993), 23-30.

[6] F. Böner, F. R. Pöschel.: Clones of operations on binary relations. Contributions to general algebras, **7** (1991), 50-70.

Dept of Mathematics, Saratov State Technical Univ., Russia

E-mail address: bredikhin@mail.ru